Part I

Answer each of the following problems in the space provided for each problem.

1. (21 points) For which values of a and b does the system

$$\begin{cases} x + 3y - z = 3 \\ 2x + 5y - az = 0 \\ 3x + 7y + 2z = b \end{cases}$$

have a) no solution, b) a unique solution, c) infinitely many solutions?

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 2 & 5 & -a & 0 \\ 3 & 7 & 2 & b \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -1 & 2-a & -6 \\ 0 & -2 & 5 & b-9 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -1 & 2-a & -6 \\ 0 & 0 & 2a+1 & b+3 \end{array}\right]$$

a) No solution:

$2a+1=0$ and $b+3 \neq 0$

$a=-\frac{1}{2}$ and $b \neq -3$

b) Unique Solution

$2a+1 \neq 0 \Rightarrow a \neq -\frac{1}{2}$

c) infinitely many solutions:

$2a+1=0$ and $b+3=0$

$a=-\frac{1}{2}$ and $b=-3$

2. (10 points) Let A be an invertible matrix such that $(A^{-1}+I)$ is invertible. Show that $(A+I)^{-1} = (A^{-1}+I)^{-1}A^{-1}$.

$$\underbrace{(A^{-1}+I)^{-1}A^{-1}}(A+I) = (A^{-1}+I)^{-1}(I+A^{-1}) = I$$

So $A+I$ has inverse $(A^{-1}+I)^{-1}A^{-1}$

3. (9 points) Show that if $A^tA = A$ then A is symmetric and $A = A^2$.

$$A^t = (A^tA)^t = A^t(A^t)^t = A^tA = A$$

So A is symmetric

$$A = A^tA = A.A = A^2$$

4. (10 points) If A and B are 4×4 matrices with $|A| = 2$ and $|B| = 9$, find $|3AB^{-1}A^t|$.

$$|3AB^{-1}A^t| = 3^4 |A| . |B^{-1}||A^t|$$
$$= 3^4 . 2 . \frac{1}{|B|} . |A|$$
$$= 3^4 . 2 . \frac{1}{9} . 2 = 36$$

5. (9 points) Let A be an $n \times n$ matrix. Show that if $(2A^2+3I)^2 = A+I$ then A is invertible.

$$(2A^2+3I)^2 = (2A^2+3I)(2A^2+3I)$$
$$= 4A^4 + 6A^2 + 6A^2 + 9I$$
$$= 4A^4 + 12A^2 + 9I$$

So $4A^4 + 12A^2 + 9I = A + I$

$$4A^4 + 12A^2 - A = -8I$$
$$A(4A^3 + 12A - I) = -8I$$

So $A.\left[-\frac{1}{8}(4A^3+12A-I)\right] = I$

So A is invertible and $A^{-1} = -\frac{1}{8}(4A^3+12A-I)$

Part II

Circle the correct answer to each of the following problems IN THE TABLE ON THE FRONT PAGE. Each correct answer is worth 5 points.

6. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$. If B is the inverse of A then the sum of the diagonal entries of B is

a) -1/3

b) 4/3

c) 1

d) -2

e) none of the above

$$\left[\underbrace{\begin{matrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{matrix}}_{A} \left| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \right.\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & [illegible] & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{1}{3} & [illegible] \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

7. If A and B are symmetric matrices then

a) AB is symmetric

b) $A + B$ is symmetric

c) AB is invertible

d) the diagonal entries of AB are all zero

e) none of the above

$$(A+B)^t = A^t + B^t = A + B$$

8. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 1$ then $\begin{vmatrix} 3a+2d & 3b+2e & 3c+2f \\ g & h & i \\ d+5g & e+5h & f+5i \end{vmatrix} =$

a) 3

b) -3

c) 15

d) -15

e) none of the above.

$$1 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \frac{1}{3}\begin{vmatrix} 3a & 3b & 3c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\overset{2R_2+R_1}{=} \frac{1}{3}\begin{vmatrix} 3a+2d & 3b+2e & 3c+2f \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\overset{5R_3+R_2}{=} \frac{1}{3}\begin{vmatrix} 3a+2d & 3b+2e & 3c+2f \\ d+5g & e+5h & f+5i \\ g & h & i \end{vmatrix}$$

$$\overset{R_2 \leftrightarrow R_3}{=} -\frac{1}{3}\begin{vmatrix} 3a+2d & 3b+2e & 3c+2f \\ g & h & i \\ d+5g & e+5h & f+5i \end{vmatrix}$$

9. If $\begin{bmatrix} a & b & b & b \\ 0 & b & 1 & c \\ 0 & 0 & c & c \end{bmatrix}$ is in row echelon form then we must have

a) $a = b = c = 0$ or ($a = 1$ and $b = c = 0$)

b) $a \in \mathbb{R}$ and $b \neq 0$ and $c \in \mathbb{R}$

c) $a \neq 0$ and $b \in \mathbb{R}$ and $c \in \mathbb{R}$

d) ($a \neq 0$ and $b = c = 0$) or ($a \neq 0$ and $b \neq 0$ and $c \in \mathbb{R}$)

e) a, b, c are all nonzero

Part III

Determine whether each of the following statements is true or false. Circle the correct answer IN THE TABLE ON THE FRONT PAGE. Each correct answer is worth 3 points.

F 10. If $A^2 = A$ then $A = 0$ or $A = I$.

F 11. If A is not invertible then the system $Ax = b$ has infinitely many solutions.

F 12. If $A + B$ is invertible then A and B are invertible. $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

F 13. If y and z are solutions of the system $Ax = b$ then any linear combination of y and z is also a solution.

F 14. If the reduced row echelon form of the augmented matrix of a linear system has a row of zeros then the system has infinitely many solutions.

T 15. The vector $\begin{bmatrix}3\\4\\6\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}2\\2\\3\end{bmatrix}$. $-1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + 2\cdot\begin{pmatrix}2\\2\\3\end{pmatrix}$

T 16. If $A = \begin{bmatrix}3 & 5 & 7 & 0\\ 1 & -4 & 9 & 1\\ 9 & 15 & 21 & 0\\ 1 & -2 & -3 & -4\end{bmatrix}$ then A^t is not invertible.

$R_3 = 3R_1$

$\Rightarrow |A| = 0$

$\Rightarrow |A^t| = 0$